Shortest path algorithms

Data Structures and Algorithms for Computational Linguistics III (ISCL-BA-07)

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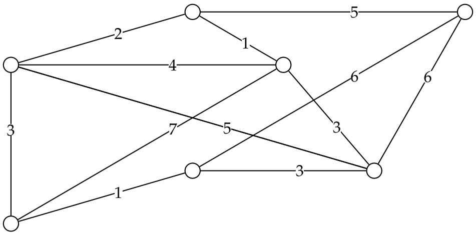
Winter Semester 2024/25

version: ab7b171 @2024-11-18

Weighted graphs

- A *weighted graph* is a graph, where each edge is associated with a weight
- Weights can be any numeric value, but some algorithms require
  - Non-negative weights
  - 'Euclidean' weights: weights that are proper distance metrics
- Weights often indicate distance or cost, but they can also represent positive relations (e.g., affinity between nodes)
- Weight of a path is the sum of wights of the edges on the path



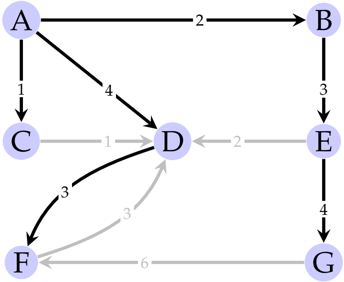
Shortest path

- Finding shortest paths on a weighted (directed) graph is one of the most common problems in many fields
- Applications include
  - Navigation
  - Routing in computer networks
  - Optimal construction of electronic circuits, VLSI chips
  - Robotics, transportation, finance, ...

Shortest paths on unweighted graphs

BFS

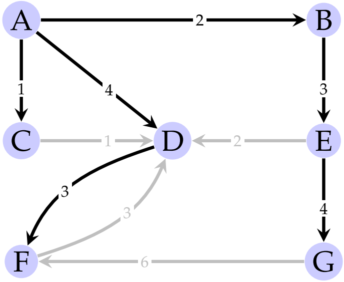
- A BFS search tree gives the shortest path from the source node to all other nodes



Shortest paths on unweighted graphs

BFS

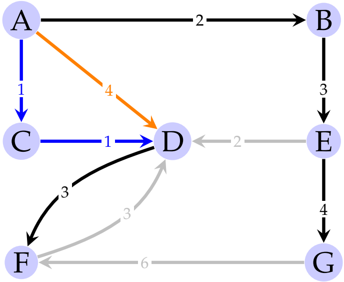
- A BFS search tree gives the shortest path from the source node to all other nodes
- The BFS is not enough on weighted graphs



Shortest paths on unweighted graphs

BFS

- A BFS search tree gives the shortest path from the source node to all other nodes
- The BFS is not enough on weighted graphs
- Shortest-cost path may be longer in terms of nodes visited



Shortest paths on weighted graphs

variations of the problem

- Different versions of the problem:
  - Single source shortest path: find shortest path from a source node to all others
  - Single target (sometimes called sink) shortest path: find shortest path from all nodes to a target node
  - Source to target: from a particular source node to a particular target node
  - All pairs: shortest paths between all pairs of nodes
- Restrictions on weights:
  - Euclidean weights
  - Non-negative weights
  - Arbitrary weights

Dijkstra's algorithm

intro

- Dijkstra's algorithm is a 'weighted' version of the BFS
- The algorithm finds shortest path from a single source node to all connected nodes
- Weights have to be non-negative
- It is a greedy algorithm that grows a 'cloud' of nodes for which we know the shortest paths from the source node
- The new nodes are included in the cloud in order of their shortest paths from the source node

Dijkstra's algorithm

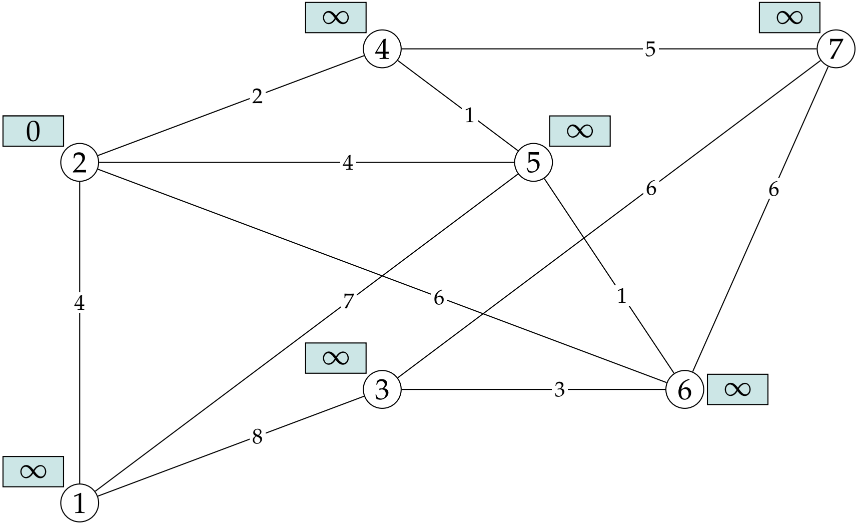
the algorithm

- We maintain a list D of minimum know distances to each node
- At each step
  - we take closest node out of Q
  - update the distances of all nodes
- Can be more efficient if Q is implemented using a (adaptable) priority queue

$D[s] \leftarrow 0$
for each node $v \neq s$ **do**
    $D[v] \leftarrow \infty$
$Q \leftarrow$ nodes
while Q is not empty **do**
    Remove node u with min D[u] from Q
    for each edge (u, v) **do**
        if $D[u] + w(u, v) < D[v]$ **then**
            $D[v] \leftarrow D[u] + w(u, v)$
D contains the shortest distances from s

Dijkstra's algorithm

demonstration



Dijkstra's algorithm

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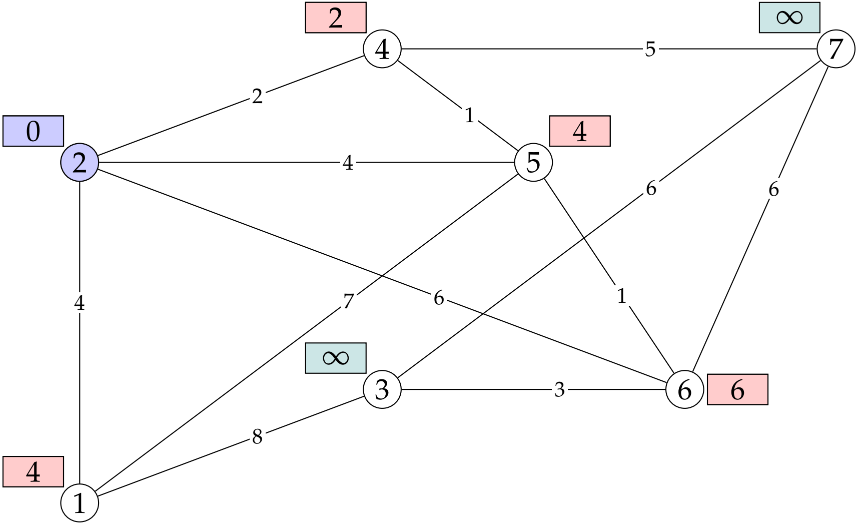

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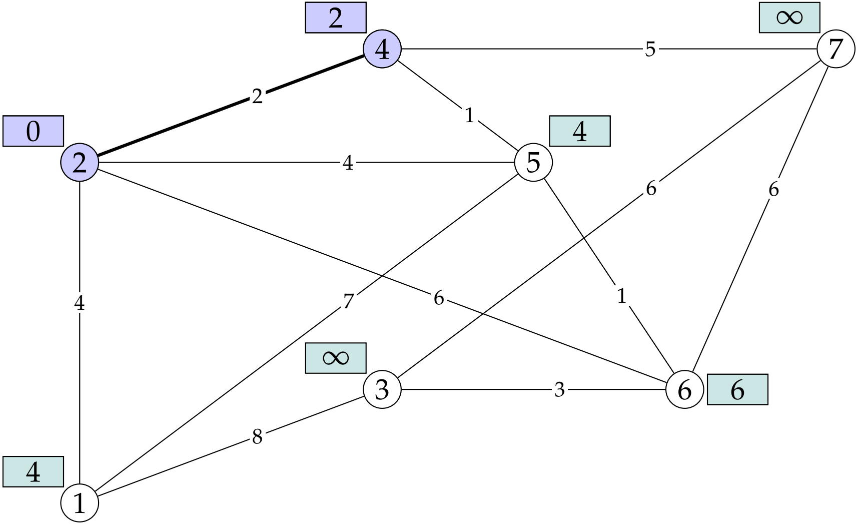

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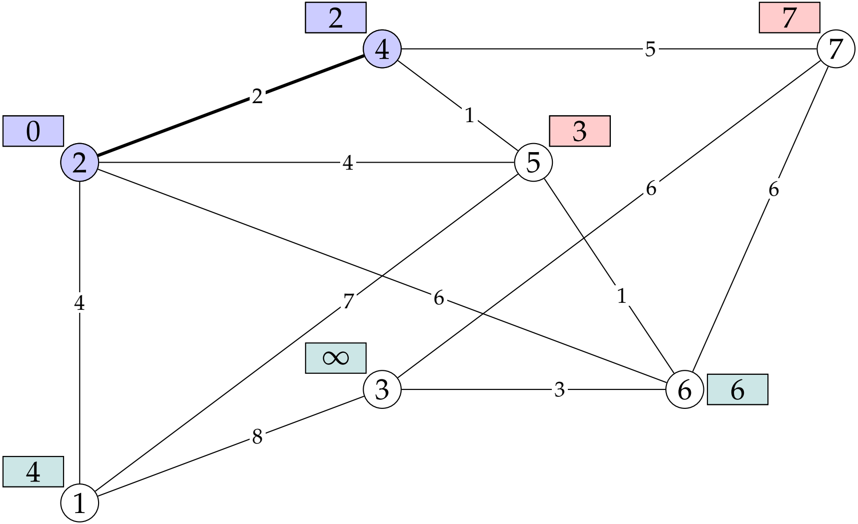

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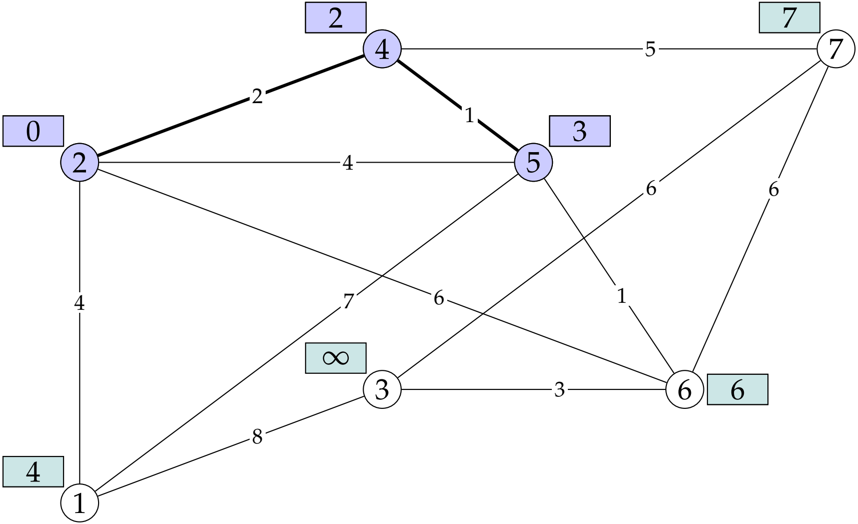

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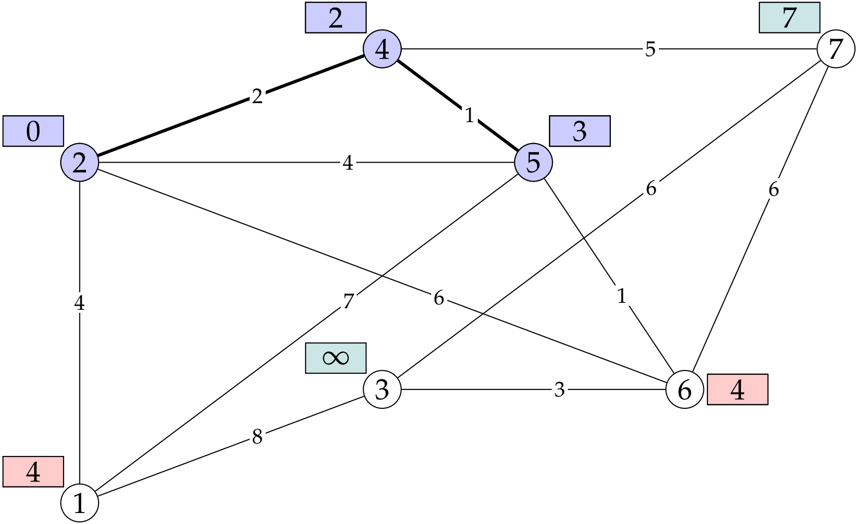

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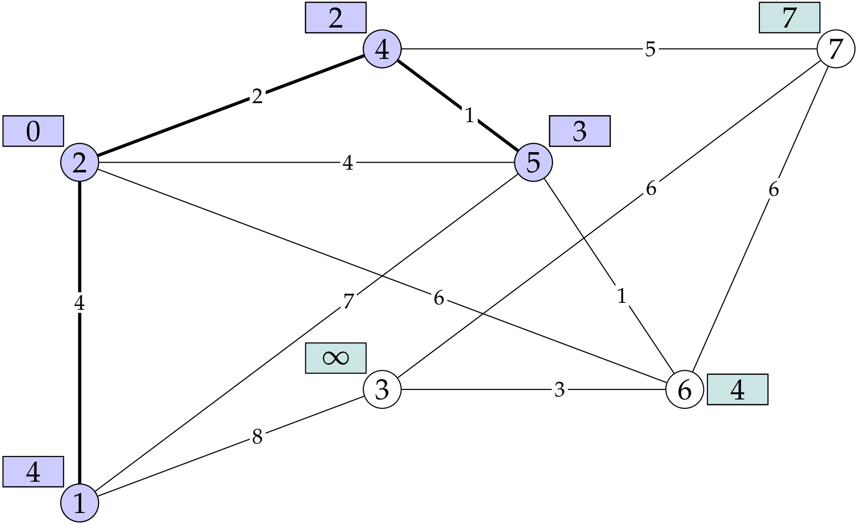

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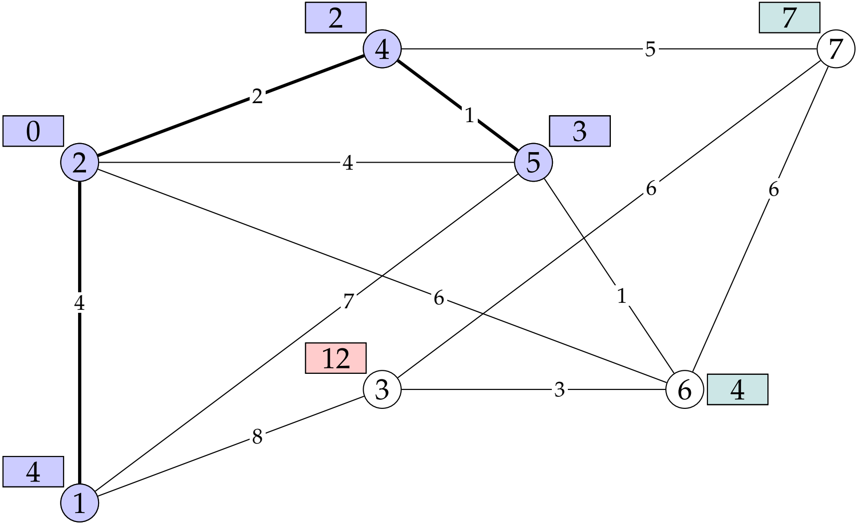

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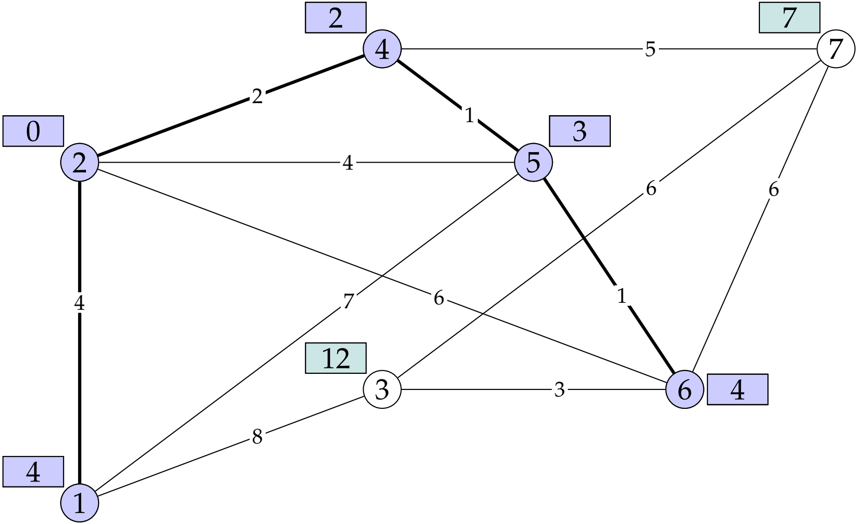

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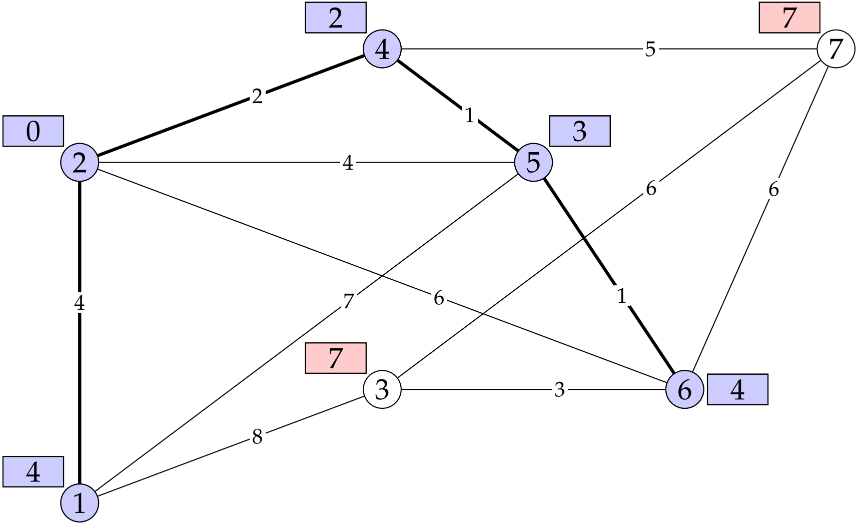

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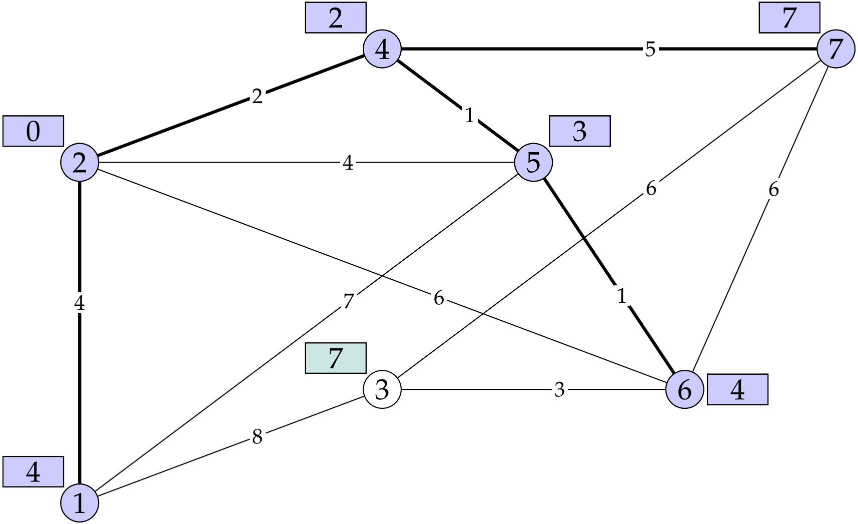

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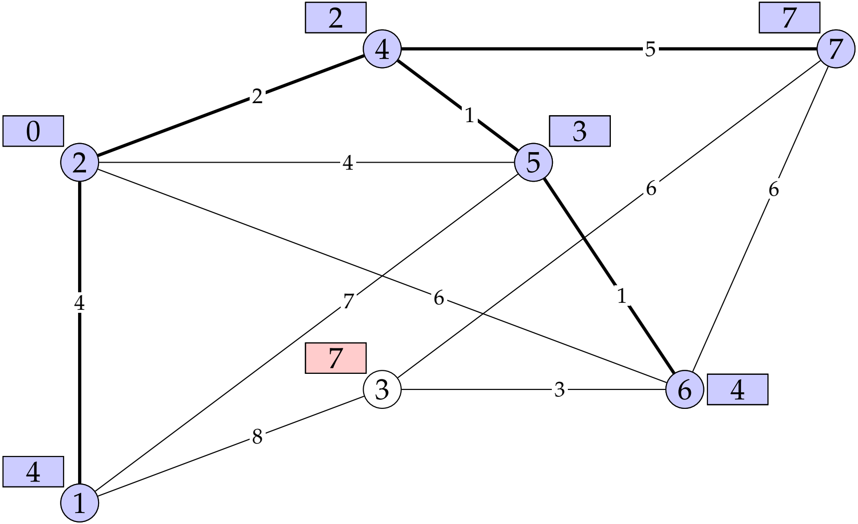

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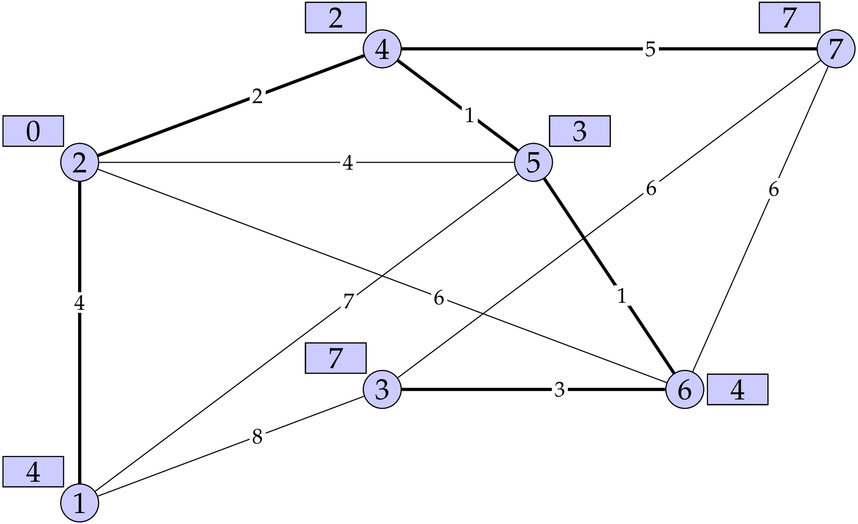
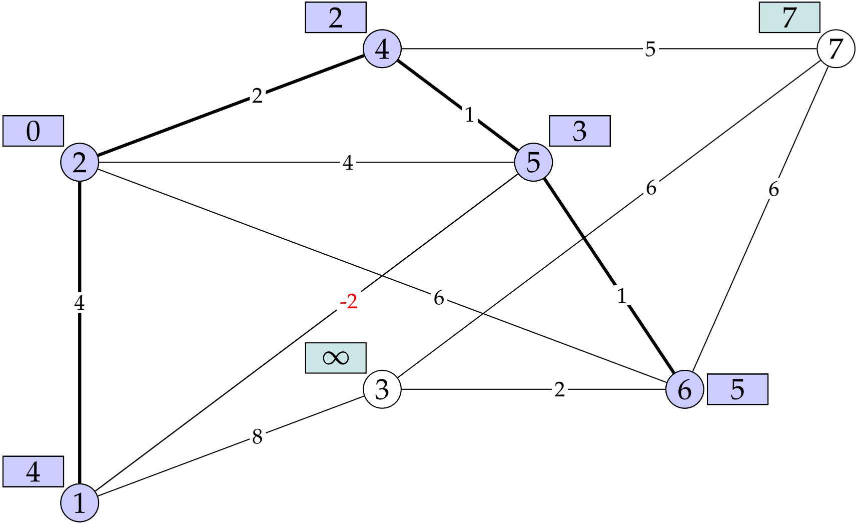


Dijkstra's algorithm and negative weights



Dijkstra's algorithm

complexity

- In general, complexity is $O(n \times t_{\text{find\_min}} + m \times t_{\text{update\_key}})$
- With list-based implementation of Q: $O(m + n^2) = O(n^2)$
- With a heap: $O((m + n) \log n)$

```
D[s] ← 0
for each node v ≠ s do
    D[v] ← ∞
Q ← nodes
while Q is not empty do
    Remove node u with min D[u] from Q
    for each edge (u, v) do
        if D[u] + w(u, v) < D[v] then
            D[v] ← D[u] + w(u, v)
D contains the shortest distances from s
```

Shortest-path tree

- The way we introduced, the Dijkstra's algorithm does not give the shortest-path tree
- Similar to traversal algorithms, we can extract it from distances D
- Running time is $O(n^2)$ (or $O(n + m)$)

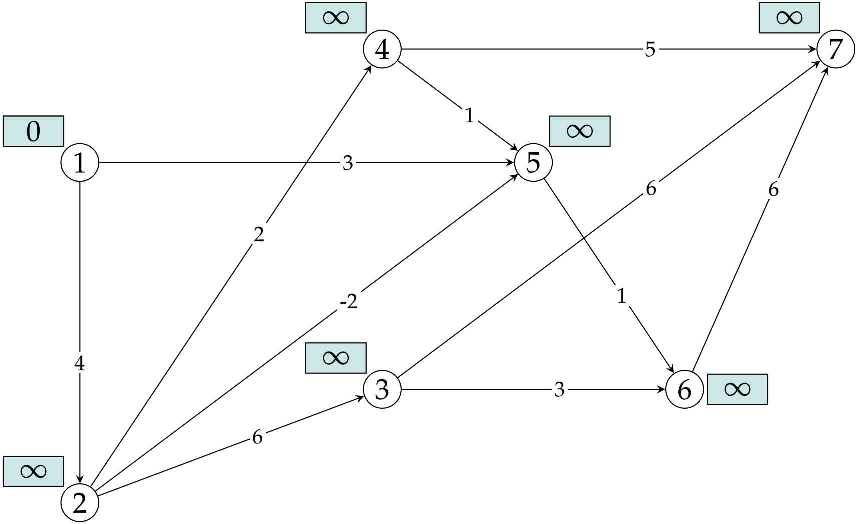
```
T ← ∅
for u ∈ D − {s} do
    for each edge (v, u) do
        if D[u] == D[v] + w(v, u) then
            T ← T ∪ (v, u)
```

Shortest-paths on DAGs

- The shortest path can be found more efficiently, if the graph is a DAG
- The algorithm is similar to Dijkstra's, but simpler and faster
- Only difference is we follow a topological order
- The algorithm will also work with negative edge weights

Shortest-paths on DAGs

demonstration



Shortest-paths on DAGs

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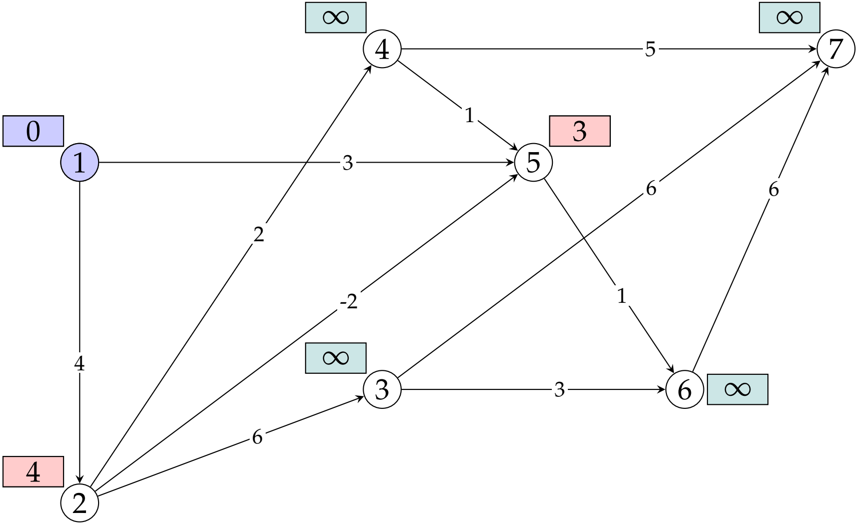

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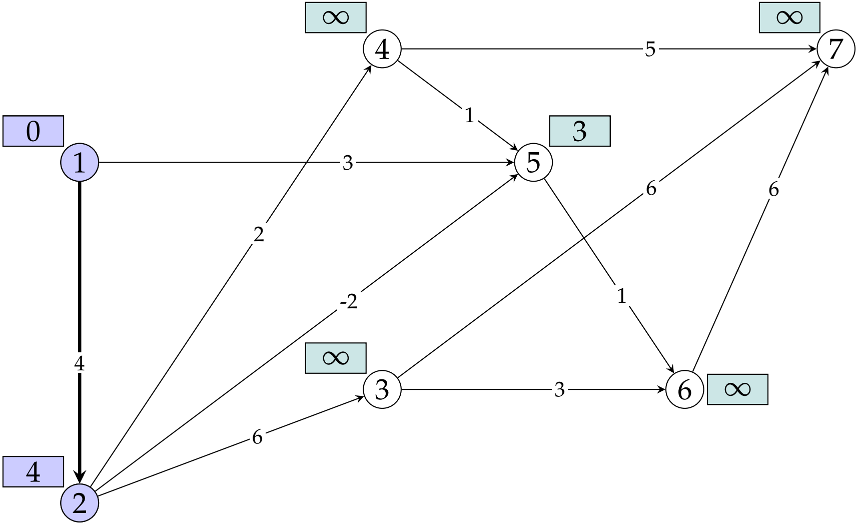

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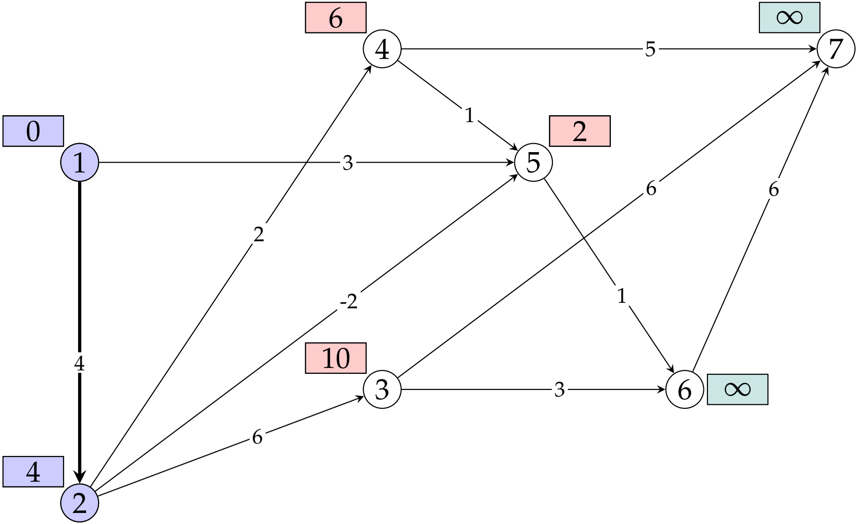

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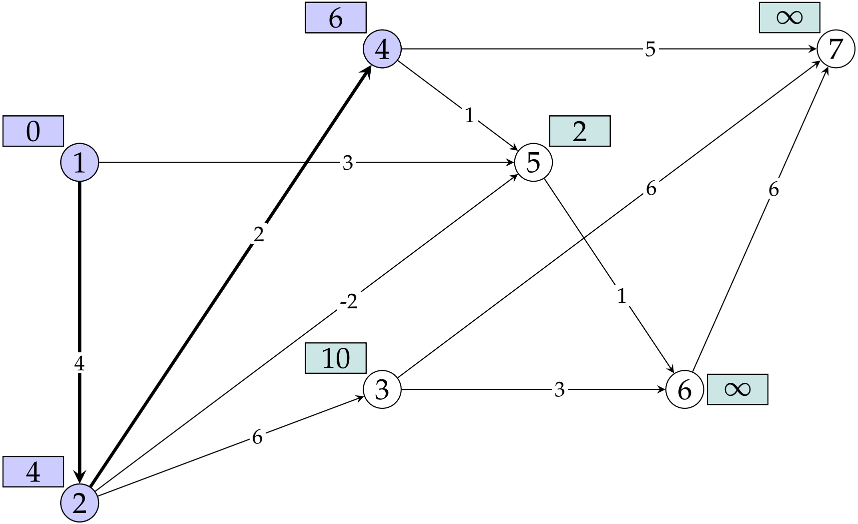

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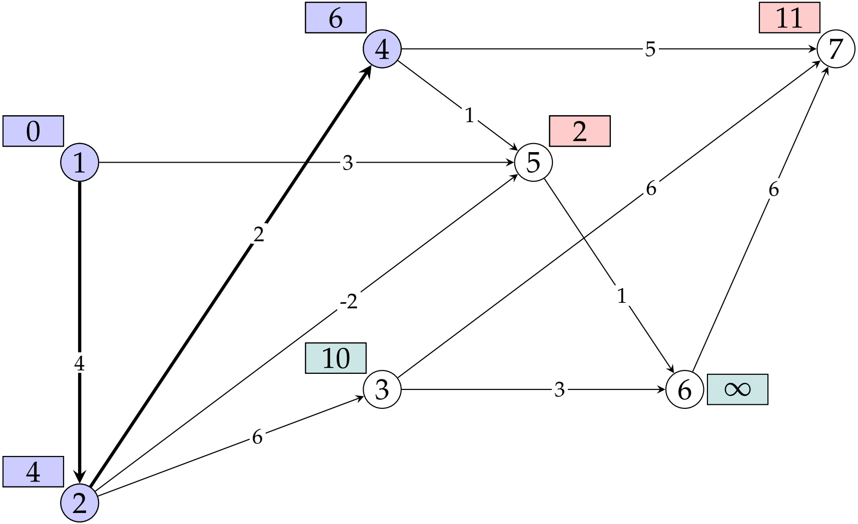

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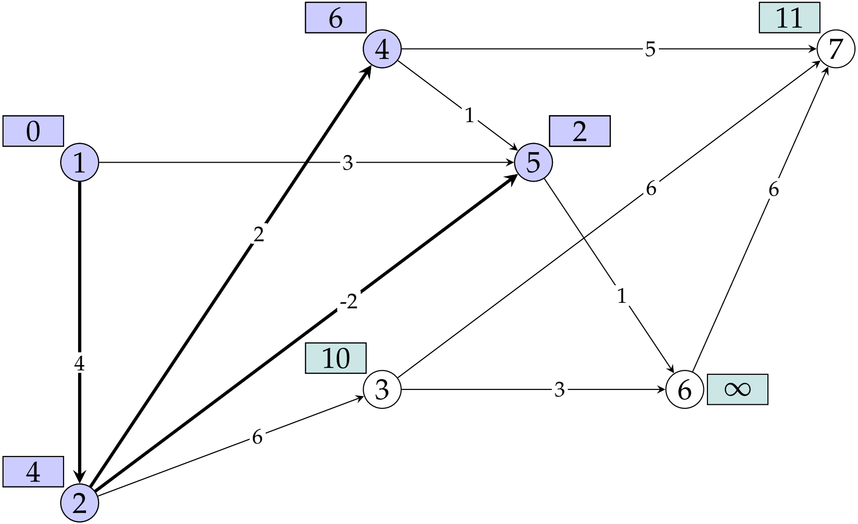

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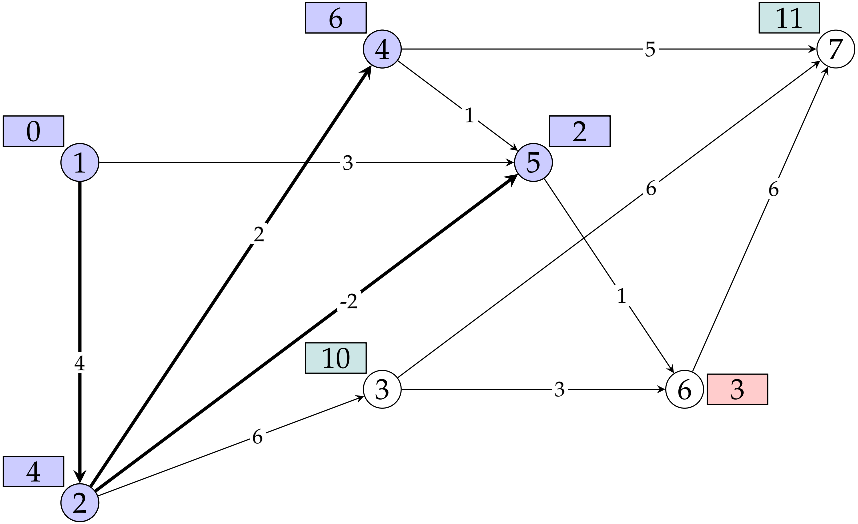

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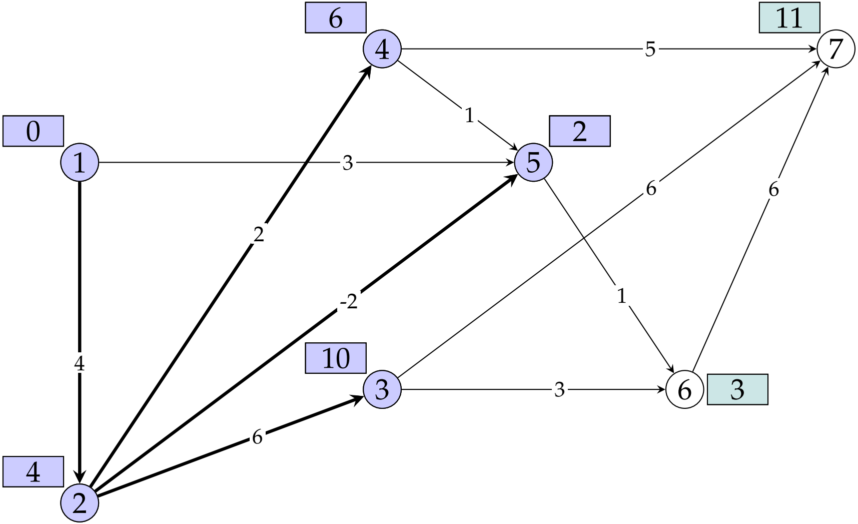

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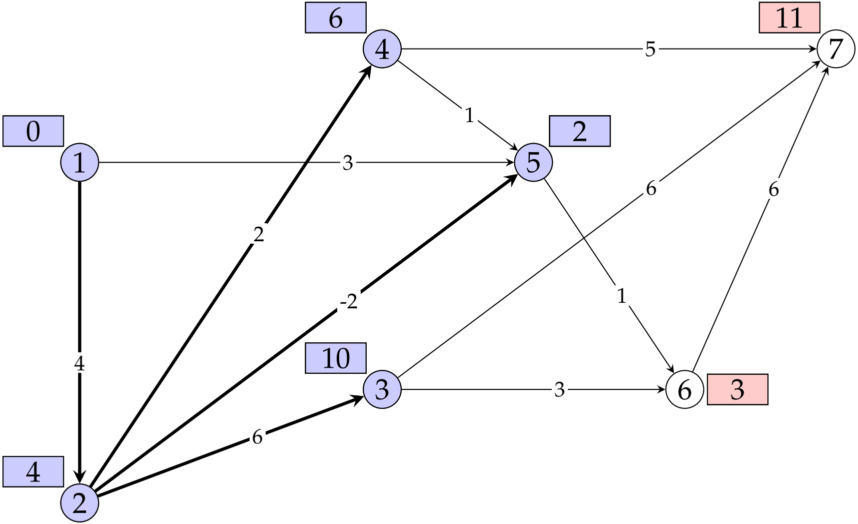

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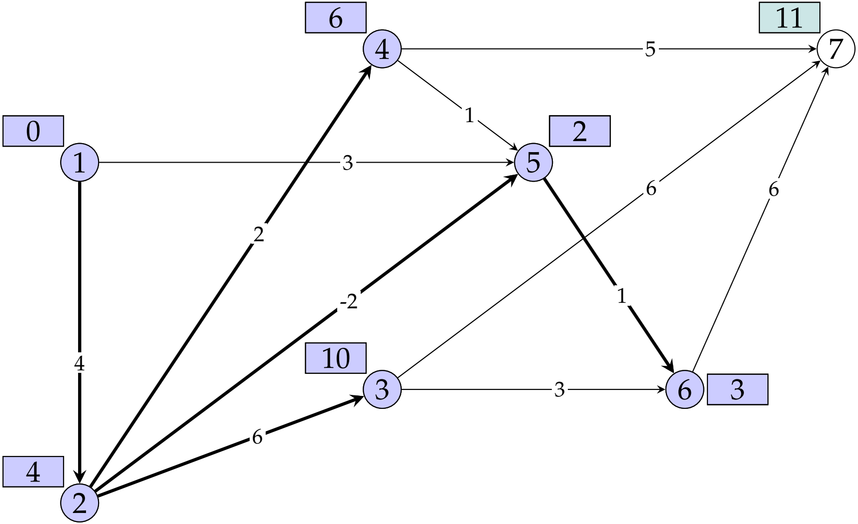

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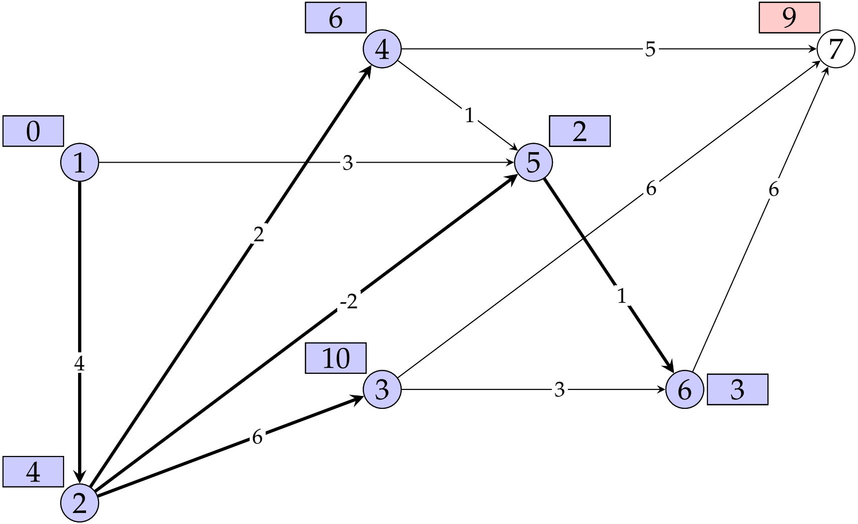

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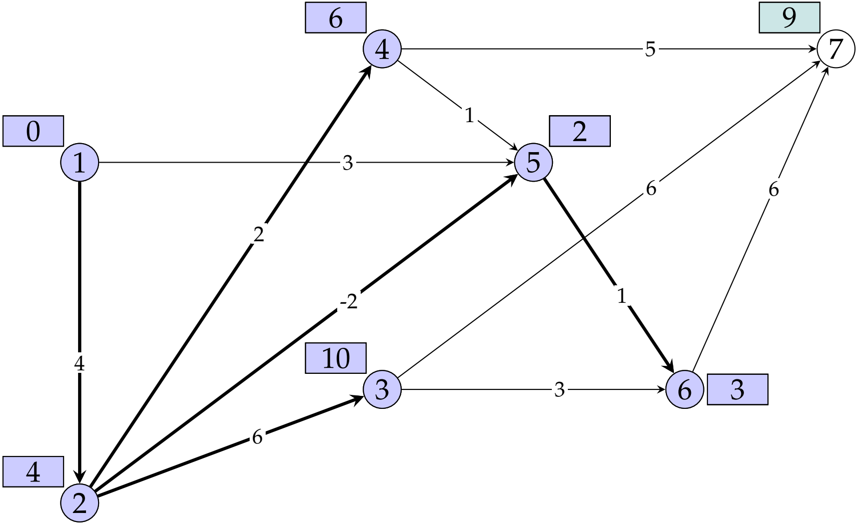

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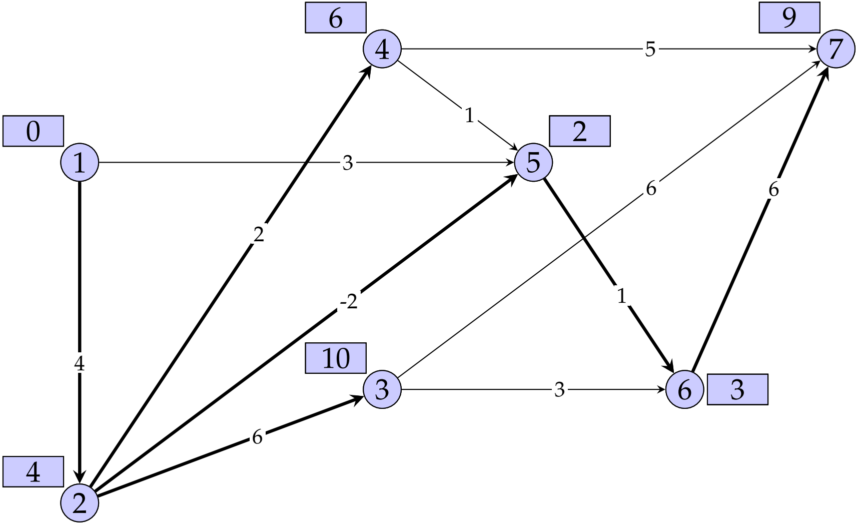


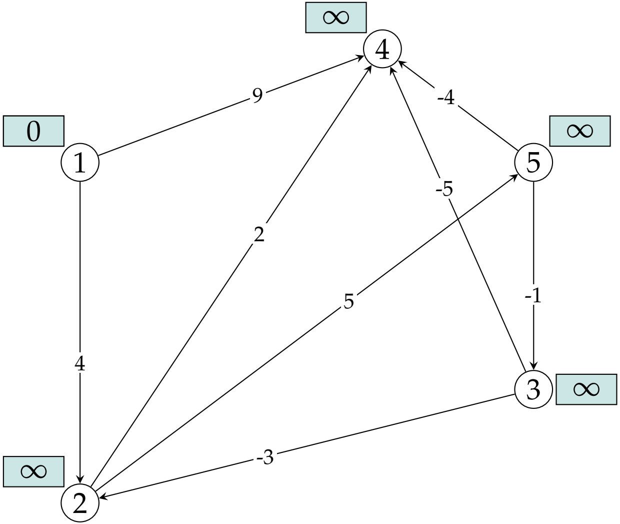
Shortest-paths on directed graphs

with negative wights – without negative cycles

- Single-source shortest path problem can also be solved efficiently for any directed graph
  - including cycles (no DAG requirement)
  - including negative weights
  - *excluding* negative cycles
- The algorithms is known as Bellman-Ford algorithm
  - Similar to earlier algorithms, initialize $D[s] = 0$, $D[v] = \infty$ for $v \neq s$
  - Make n passes over the edges
    - Update distances for each edge (relax edges)
    - Stop if there were no changes at the end of a pass

Bellman-Ford algorithm

demonstration



edges

$1 \rightarrow 2$
$1 \rightarrow 4$
$2 \rightarrow 4$
$2 \rightarrow 5$
$3 \rightarrow 2$
$3 \rightarrow 4$
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$5 \rightarrow 4$

Bellman-Ford algorithm

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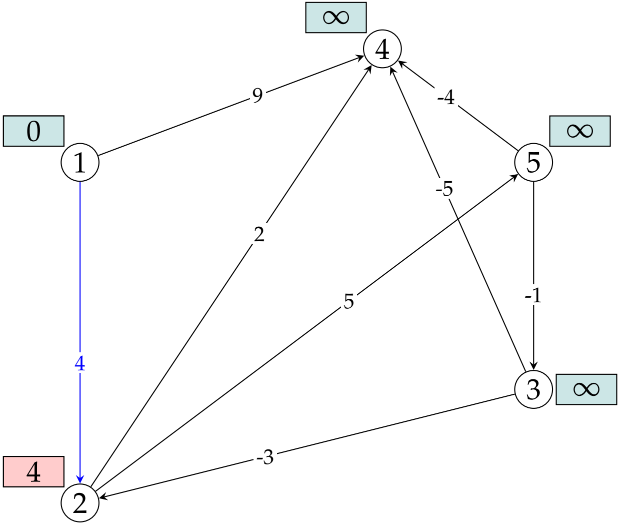
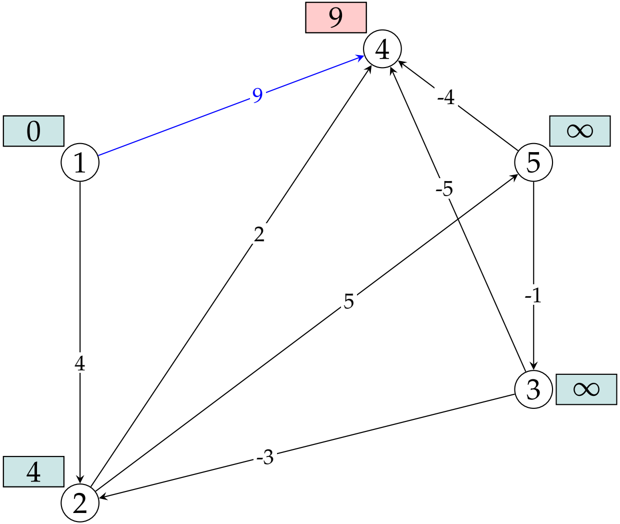


edges

1 → 2
1 → 4
2 → 4
2 → 5
3 → 2
3 → 4
5 → 3
5 → 4

Bellman-Ford algorithm

demonstration



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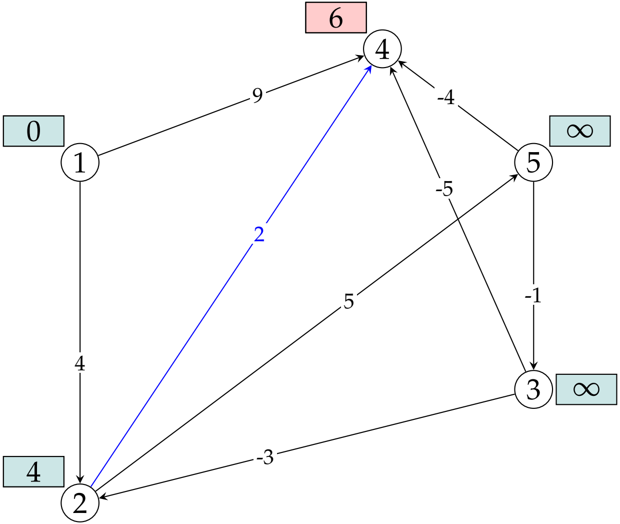

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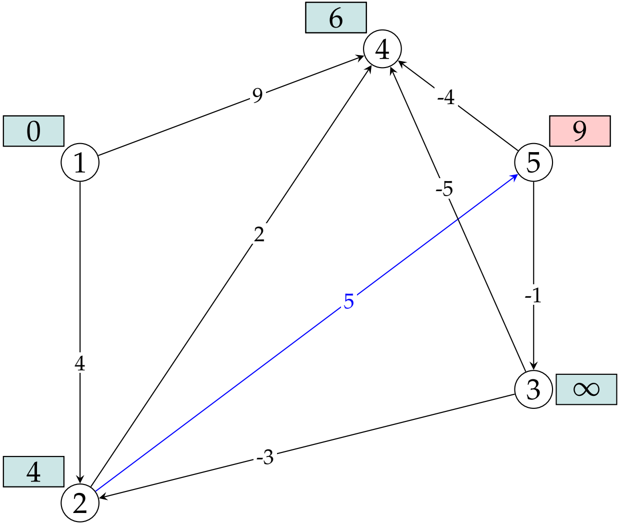

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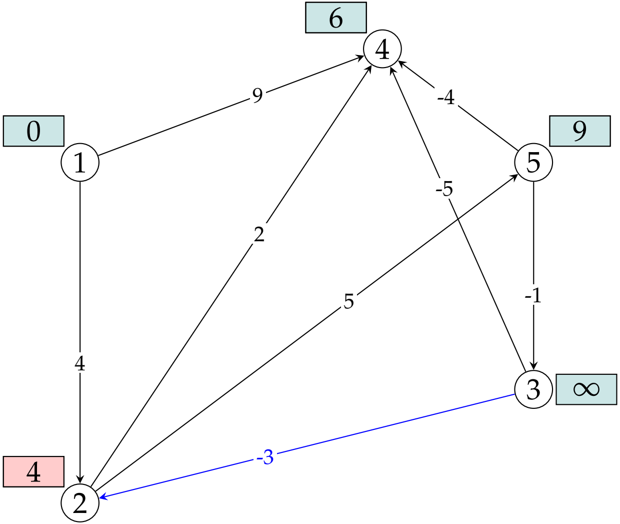

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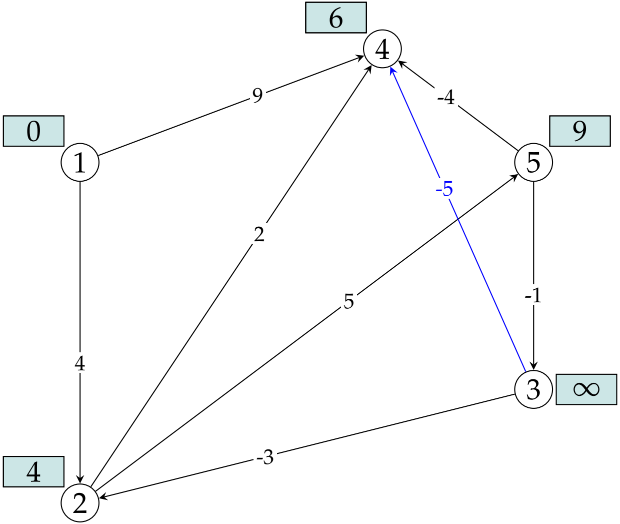

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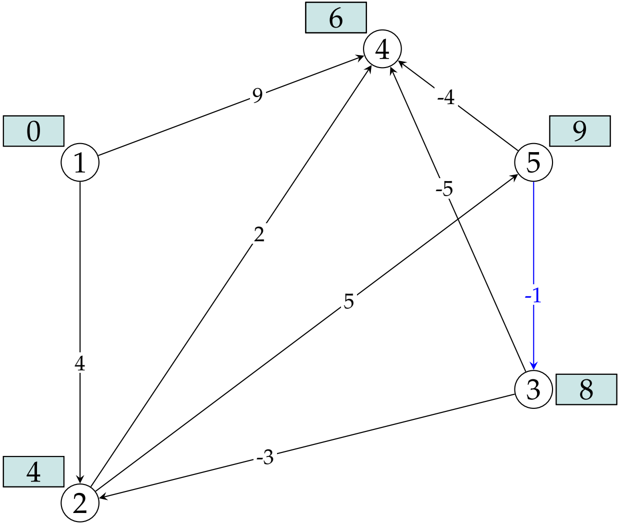

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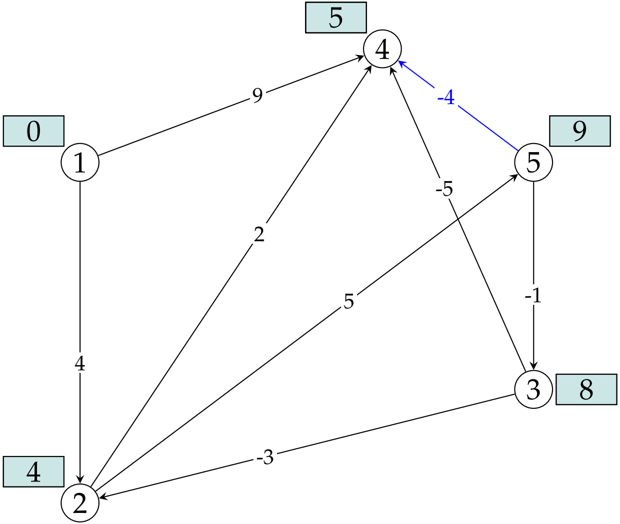

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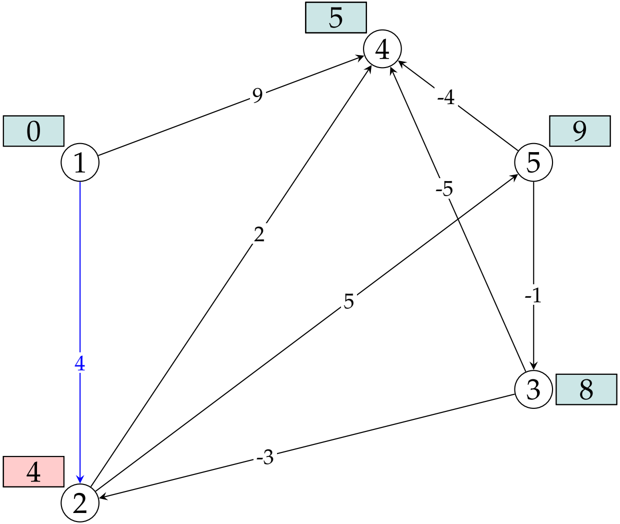

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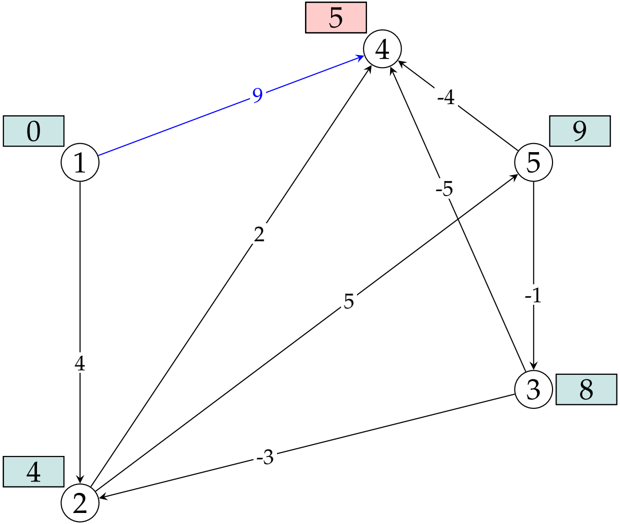

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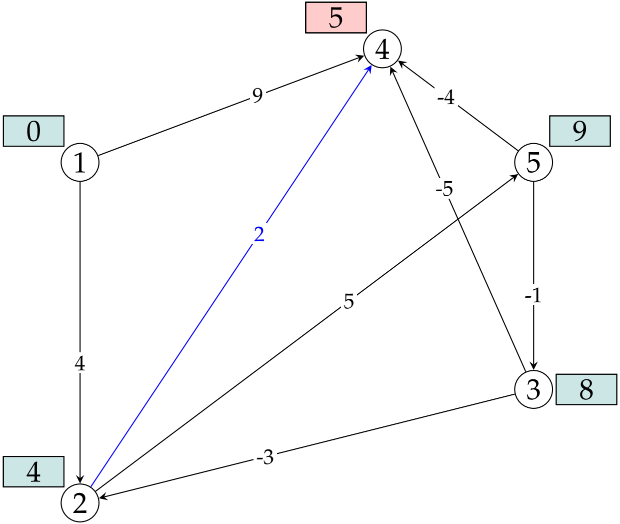

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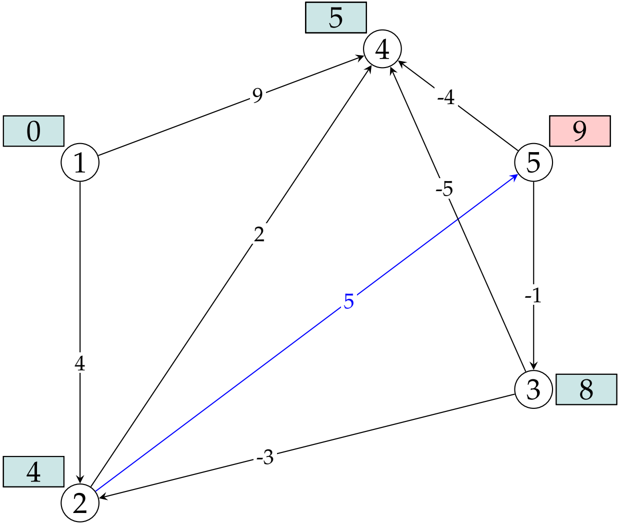

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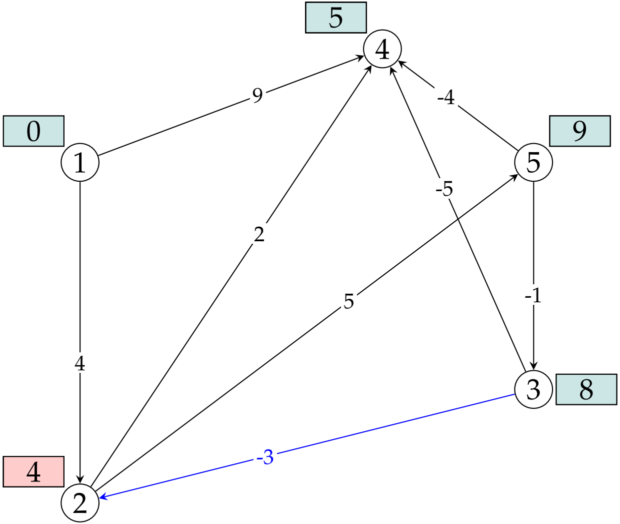

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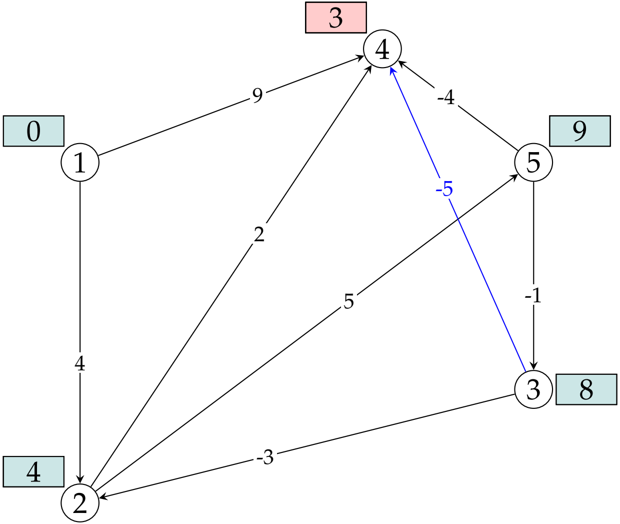

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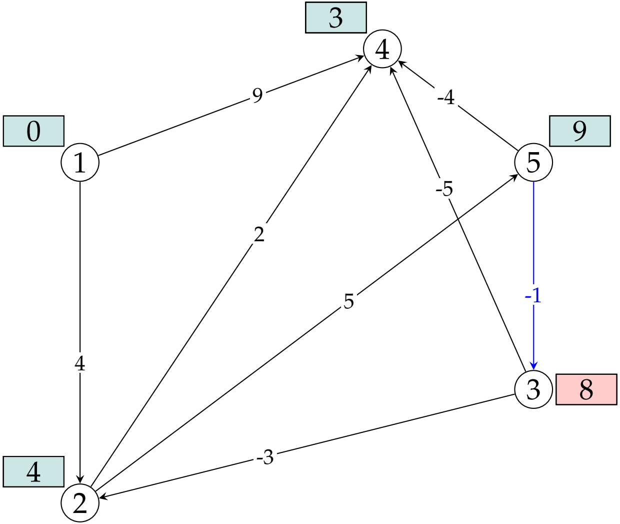

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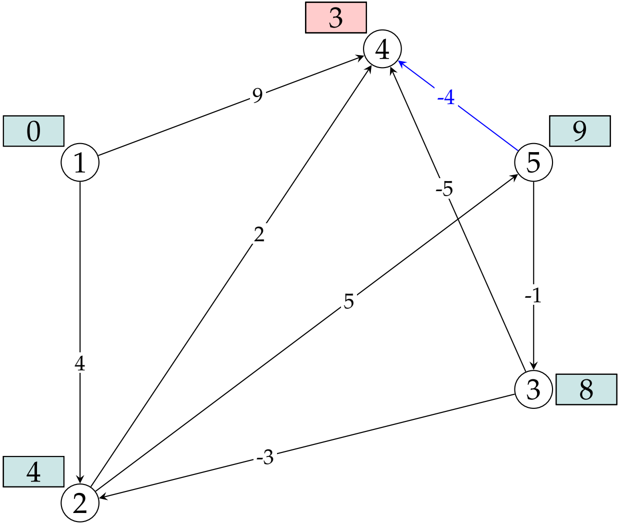

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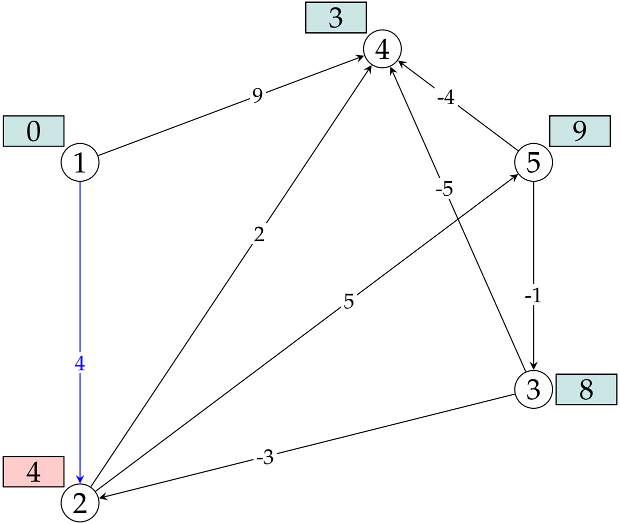
Bellman-Ford algorithm

demonstration

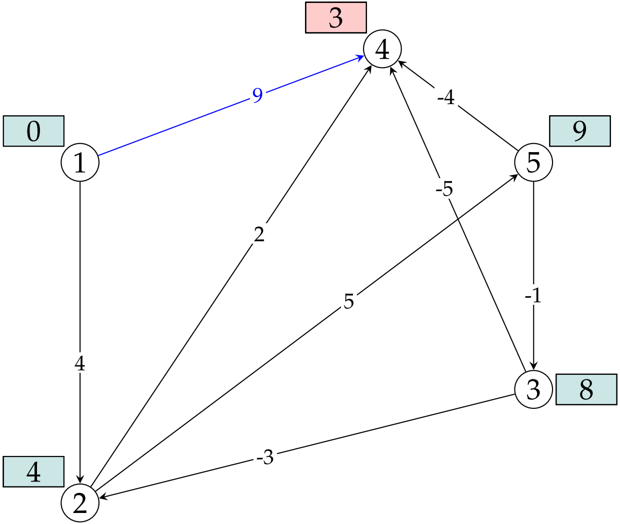


edges

$1 \rightarrow 2$
$1 \rightarrow 4$
$2 \rightarrow 4$
$2 \rightarrow 5$
$3 \rightarrow 2$
$3 \rightarrow 4$
$5 \rightarrow 3$
$5 \rightarrow 4$

Bellman-Ford algorithm

demonstration

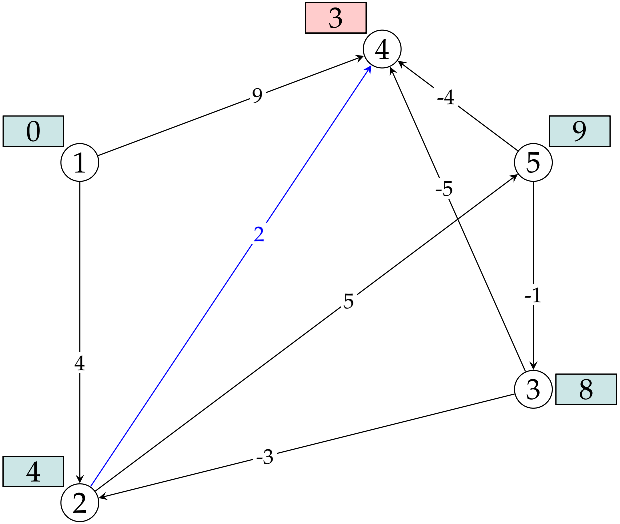


edges

1 → 2
1 → 4
2 → 4
2 → 5
3 → 2
3 → 4
5 → 3
5 → 4

Bellman-Ford algorithm

demonstration

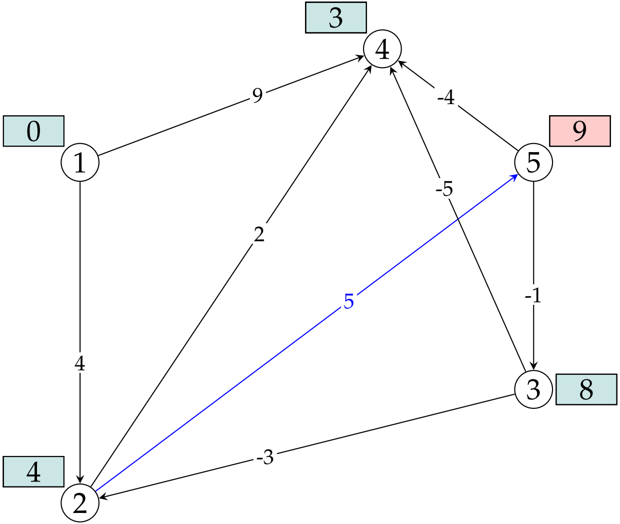


edges

$1 \rightarrow 2$
$1 \rightarrow 4$
$2 \rightarrow 4$
$2 \rightarrow 5$
$3 \rightarrow 2$
$3 \rightarrow 4$
$5 \rightarrow 3$
$5 \rightarrow 4$

Bellman-Ford algorithm

demonstration

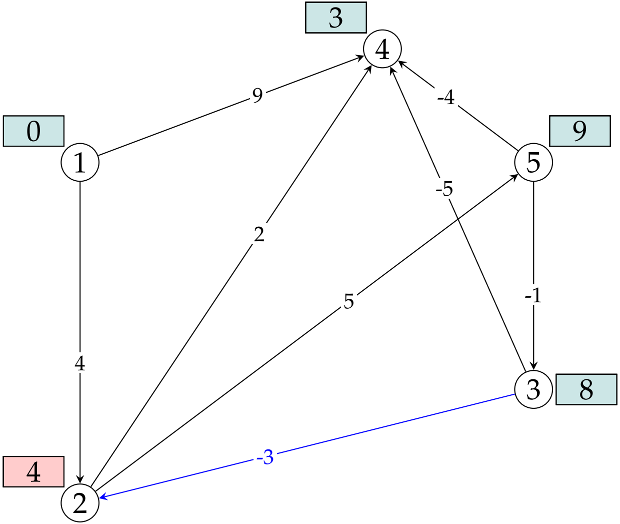


edges

1 → 2
1 → 4
2 → 4
2 → 5
3 → 2
3 → 4
5 → 3
5 → 4

Bellman-Ford algorithm

demonstration

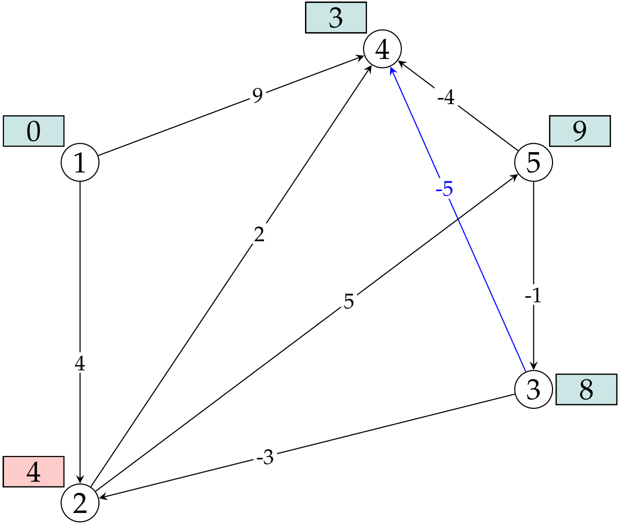


edges

$1 \rightarrow 2$
$1 \rightarrow 4$
$2 \rightarrow 4$
$2 \rightarrow 5$
$3 \rightarrow 2$
$3 \rightarrow 4$
$5 \rightarrow 3$
$5 \rightarrow 4$

Bellman-Ford algorithm

demonstration

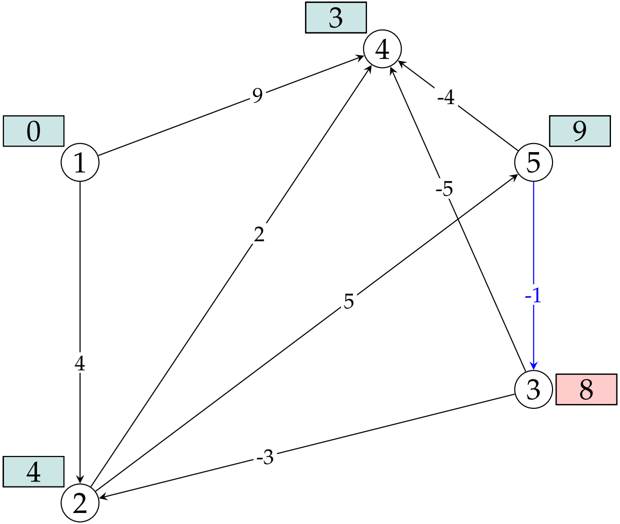


edges

$1 \rightarrow 2$
$1 \rightarrow 4$
$2 \rightarrow 4$
$2 \rightarrow 5$
$3 \rightarrow 2$
$3 \rightarrow 4$
$5 \rightarrow 3$
$5 \rightarrow 4$

Bellman-Ford algorithm

demonstration



edges

$1 \rightarrow 2$
$1 \rightarrow 4$
$2 \rightarrow 4$
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$3 \rightarrow 2$
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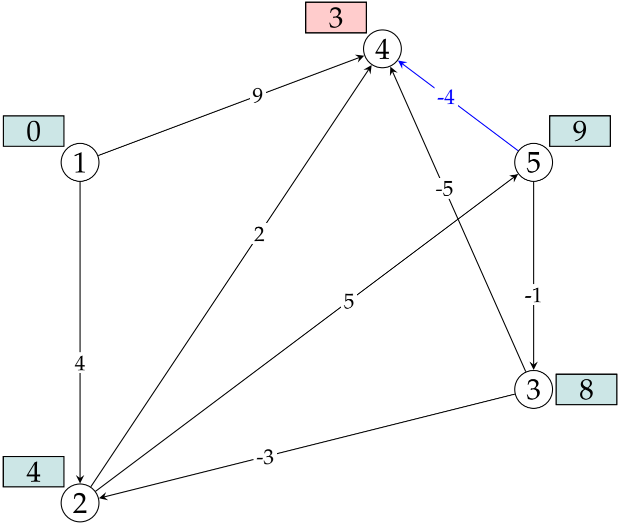
Bellman-Ford algorithm

demonstration



edges

1 → 2
1 → 4
2 → 4
2 → 5
3 → 2
3 → 4
5 → 3
5 → 4

Summary

- Shortest path algorithms are one of the most applied graph algorithms
- We revised three algorithms
  - Dijkstra's: non-negative weights, general algorithm
  - For DAGs: unrestricted weights, following topological order
  - Bellman-Ford: no negative cycles, digraphs
- Reading: Goodrich, Tamassia, and Goldwasser (2013, chapter 14)

Next:

- Minimum spanning trees
- Reading: Goodrich, Tamassia, and Goldwasser (2013, chapter 14)

Acknowledgments, credits, references

Goodrich, Michael T., Roberto Tamassia, and Michael H. Goldwasser (2013). *Data Structures and Algorithms in Python*. John Wiley & Sons, Incorporated. ISBN: 9781118476734.

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