Slide 1

Graphs
Data Structures and Algorithms for Computational Linguistics III
(ISCL-BA-07)

Çağrı Çöltekin
ccoltekin@sfs.uni-tuebingen.de

University of Tübingen
Seminar für Sprachwissenschaft

Winter Semester 2024/25

Slide 2

Introduction

- A graph is collection of **vertices (nodes)** connected pairwise by **edges (arcs)**.
- A graph is a useful abstraction with many applications
- Most problems on graphs are challenging

Slide 3

Example applications
City map

- **City maps**
- Chemical formulas
- Neural networks
- Artificial neural networks
- Electronic circuits
- Computer networks
- Infectious diseases
- Probability distributions
- Word semantics

Slide 4

Example applications
City map

- City maps
- **Chemical formulas**
- Neural networks
- Artificial neural networks
- Electronic circuits
- Computer networks
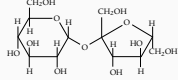- Infectious diseases
- Probability distributions
- Word semantics

Slide 5

Example applications
City map

- City maps
- Chemical formulas
- **Neural networks**
- Artificial neural networks
- Electronic circuits
- Computer networks
- Infectious diseases
- Probability distributions
- Word semantics

Slide 6

Example applications
City map

- City maps
- Chemical formulas
- Neural networks
- **Artificial neural networks**
- Electronic circuits
- Computer networks
- Infectious diseases
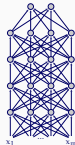- Probability distributions
- Word semantics

Slide 7

Example applications
City map

- City maps
- Chemical formulas
- Neural networks
- Artificial neural networks
- **Electronic circuits**
- Computer networks
- Infectious diseases
- Probability distributions
- Word semantics
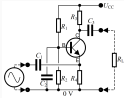

Slide 8

Example applications
City map

- City maps
- Chemical formulas
- Neural networks
- Artificial neural networks
- Electronic circuits
- **Computer networks**
- Infectious diseases
- Probability distributions
- Word semantics

Slide 9

Example applications
City map

- City maps
- Chemical formulas
- Neural networks
- Artificial neural networks
- Electronic circuits
- Computer networks
- **Infectious diseases**
- Probability distributions
- Word semantics

Slide 10

Example applications
City map

- City maps
- Chemical formulas
- Neural networks
- Artificial neural networks
- Electronic circuits
- Computer networks
- Infectious diseases
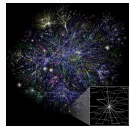- **Probability distributions**
- Word semantics

Slide 11
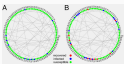
Example applications
City map

- City maps
- Chemical formulas
- Neural networks
- Artificial neural networks
- Electronic circuits
- Computer networks
- Infectious diseases
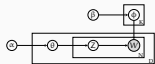- Probability distributions
- **Word semantics**

Slide 12

Example applications
many more...

- Food web
- Course dependencies
- Social media
- Scheduling
- Games
- Academic networks
- Inheritance relations in object-oriented programming
- Flow charts
- Financial transactions
- World's languages
- PageRank algorithm
- ...

Definition

- A (simple) *graph* G is a pair (V, E) where
  - V is a set of *nodes* (or vertices),
  - E ⊆ {(x, y) | x, y ∈ V and x ≠ y} is a set of ordered or unordered pairs, *edges*
- A graph represent a set of objects (nodes) and the relations between them (edges)
- Edges in a graph can be either *directed*, or *undirected*
  - directed edges (also called *arcs*) are 2-tuples, or *ordered pairs* (order is important)
  - undirected edges are unordered pairs, or pair sets (order is not important)

Types of graphs

- An *undirected graph* is a graph with only undirected edges
  - Transportation (e.g., railway) networks
- A *directed graph* (digraph) is a graph with only directed edges
  - course dependencies
- A *mixed graph* contains both directed and undirected edges
  - a city map

Types of graphs

- An *undirected graph* is a graph with only undirected edges
  - Transportation (e.g., railway) networks
- A *directed graph* (digraph) is a graph with only directed edges
  - course dependencies
- A *mixed graph* contains both directed and undirected edges
  - a city map

Types of graphs

- An *undirected graph* is a graph with only undirected edges
  - Transportation (e.g., railway) networks
- A *directed graph* (digraph) is a graph with only directed edges
  - course dependencies
- A *mixed graph* contains both directed and undirected edges
  - a city map

More graphs types

- A graph is *simple* if there is only a single edge between two nodes (our earlier definition)
- If the edges of a graph has associated weights, it is called a *weighted graph*
- A *complete graph* contains edges from each node to every other node
- A *bipartite graph* has two disjoint sets of nodes, where edges are always across the sets
- A graph is called a *multi-graph* if there are multiple edges (with the same direction) between a pair of nodes
- A graph is called a *hyper-graph* if a single edge can link more than two nodes

---

More definitions

- Two nodes joined by an edge are called the *endpoints* of the edge
- An edge is called *incident* to a node if the node is one of its endpoints. Two nodes are *adjacent* (or they are *neighbors*) if they are incident to the same edge
- The *degree* (or valency) of a node is the number of its incident edges
- In a digraph *indegree* of a node is the number of incoming edges, and *outdegree* a node is the number of outgoing edges

A and B are endpoints of edge 1

More definitions

- Two nodes joined by an edge are called the *endpoints* of the edge
- An edge is called *incident* to a node if the node is one of its endpoints. Two nodes are *adjacent* (or they are *neighbors*) if they are incident to the same edge
- The *degree* (or valency) of a node is the number of its incident edges
- In a digraph *indegree* of a node is the number of incoming edges, and *outdegree* a node is the number of outgoing edges

edge 1 is incident to A and B

More definitions

- Two nodes joined by an edge are called the *endpoints* of the edge
- An edge is called *incident* to a node if the node is one of its endpoints. Two nodes are *adjacent* (or they are *neighbors*) if they are incident to the same edge
- The *degree* (or valency) of a node is the number of its incident edges
- In a digraph *indegree* of a node is the number of incoming edges, and *outdegree* a node is the number of outgoing edges

deg(A) = 4

More definitions

- Two nodes joined by an edge are called the *endpoints* of the edge
- An edge is called *incident* to a node if the node is one of its endpoints. Two nodes are *adjacent* (or they are *neighbors*) if they are incident to the same edge
- The *degree* (or valency) of a node is the number of its incident edges
- In a digraph *indegree* of a node is the number of incoming edges, and *outdegree* a node is the number of outgoing edges

indeg(A) = 1, outdeg(A) = 3

More definitions

- Two edges are *parallel* if their both endpoints are the same
- For a directed graph parallel edges are ones with the same direction
- A self-loop is an edge from a node to itself
- A *path* is a sequence of alternating edges and nodes
- A *cycle* is a path that starts and ends at the same node
- A path or a cycle is a *simple* if every node on the path is visited only once

---

More definitions

- Two edges are *parallel* if their both endpoints are the same
- For a directed graph parallel edges are ones with the same direction
- A self-loop is an edge from a node to itself
- A *path* is a sequence of alternating edges and nodes
- A *cycle* is a path that starts and ends at the same node
- A path or a cycle is a *simple* if every node on the path is visited only once

---

More definitions

- Two edges are *parallel* if their both endpoints are the same
- For a directed graph parallel edges are ones with the same direction
- A self-loop is an edge from a node to itself
- A *path* is a sequence of alternating edges and nodes
- A *cycle* is a path that starts and ends at the same node
- A path or a cycle is a *simple* if every node on the path is visited only once

---

More definitions

- Two edges are *parallel* if their both endpoints are the same
- For a directed graph parallel edges are ones with the same direction
- A self-loop is an edge from a node to itself
- A *path* is a sequence of alternating edges and nodes
- A *cycle* is a path that starts and ends at the same node
- A path or a cycle is a *simple* if every node on the path is visited only once

More definitions

- Two edges are *parallel* if their both endpoints are the same
- For a directed graph parallel edges are ones with the same direction
- A self-loop is an edge from a node to itself
- A *path* is a sequence of alternating edges and nodes
- A *cycle* is a path that starts and ends at the same node
- A path or a cycle is a *simple* if every node on the path is visited only once

---

More definitions

- Two edges are *parallel* if their both endpoints are the same
- For a directed graph parallel edges are ones with the same direction
- A self-loop is an edge from a node to itself
- A *path* is a sequence of alternating edges and nodes
- A *cycle* is a path that starts and ends at the same node
- A path or a cycle is a *simple* if every node on the path is visited only once
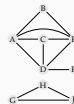

More definitions

- Two edges are *parallel* if their both endpoints are the same
- For a directed graph parallel edges are ones with the same direction
- A self-loop is an edge from a node to itself
- A *path* is a sequence of alternating edges and nodes
- A *cycle* is a path that starts and ends at the same node
- A path or a cycle is a *simple* if every node on the path is visited only once
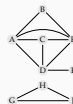

More definitions

- A node X is *reachable* from another (Y) if there is a (directed) path from Y to X
- A graph is *connected* if all nodes are reachable from each other
- A directed graph is *strongly connected* if all nodes are reachable from each other
- A *subgraph* a graph formed by a subset of nodes and edges of a graph
- If a graph is not connected, the maximally connected subgraphs are called the connected components

---

More definitions

- A node X is *reachable* from another (Y) if there is a (directed) path from Y to X
- A graph is *connected* if all nodes are reachable from each other
- A directed graph is *strongly connected* if all nodes are reachable from each other
- A *subgraph* a graph formed by a subset of nodes and edges of a graph
- If a graph is not connected, the maximally connected subgraphs are called the connected components
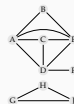

More definitions

- A node X is *reachable* from another (Y) if there is a (directed) path from Y to X
- A graph is *connected* if all nodes are reachable from each other
- A directed graph is *strongly connected* if all nodes are reachable from each other
- A *subgraph* a graph formed by a subset of nodes and edges of a graph
- If a graph is not connected, the maximally connected subgraphs are called the connected components
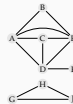

More definitions

- A node X is *reachable* from another (Y) if there is a (directed) path from Y to X
- A graph is *connected* if all nodes are reachable from each other
- A directed graph is *strongly connected* if all nodes are reachable from each other
- A *subgraph* a graph formed by a subset of nodes and edges of a graph
- If a graph is not connected, the maximally connected subgraphs are called the connected components
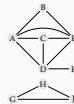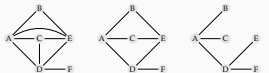

More definitions

- A node X is *reachable* from another (Y) if there is a (directed) path from Y to X
- A graph is *connected* if all nodes are reachable from each other
- A directed graph is *strongly connected* if all nodes are reachable from each other
- A *subgraph* a graph formed by a subset of nodes and edges of a graph
- If a graph is not connected, the maximally connected subgraphs are called the connected components

---

More definitions

- A *spanning subgraph* of a graph is a subgraph that includes all nodes of the graph
- A *tree* is a connected graph without cycles
- A *spanning tree* is a spanning subgraph which is a tree
- A *forest* is a disconnected acyclic graph

More definitions

- A *spanning subgraph* of a graph is a subgraph that includes all nodes of the graph
- A *tree* is a connected graph without cycles
- A *spanning tree* is a spanning subgraph which is a tree
- A *forest* is a disconnected acyclic graph

Some properties
sum of degrees

- For an undirected graph with m edges and set of nodes V

$$\sum_{v \in V} \deg(v) = 2m$$

- All edges are counted twice for each node they are incident to
- The total contribution of each node is twice its degree
- For a directed graph with m edges and set of nodes V

$$\sum_{v \in V} \text{indeg}(v) = \sum_{v \in V} \text{outdeg}(v) = m$$

Some properties
relation between the number of edges and nodes

- For a simple undirected graph with n nodes and m edges

$$m \leq \frac{n(n-1)}{2}$$

- If the graph is simple
  - there are no parallel edges
  - there are no self loops
  - the maximum degree of a node is $n - 1$
- Putting this together with the previous property

$$2m \leq n(n-1) \Rightarrow m \leq \frac{n(n-1)}{2}$$

- For a directed graph with n nodes and m edges

$$m \leq n(n-1)$$

The graph ADT

- A graph is a collection of nodes and edges
- Basic operations include
  - add_node(v) add a new node
  - remove_node(v) remove an existing node
  - adjacent(u,v) return true if the nodes are adjacent (for a digraph true only if there is a directed link from u to v)
  - neighbors(v) enumerate the neighbors of the node (for a digraph we list the nodes reachable through outgoing edges by default)
  - remove_edge(u,v) remove an existing edge
  - add_edge(u,v) add a new edge
  - nodes() enumerate the nodes in the graph
  - edges() enumerate the edges in the graph

Edge list

| e = (A,B) |
| f = (B,C) |
| g = (A,D) |
| h = (D,B) |
| k = (A,C) |

- We keep a simple a simple list of edges (and possibly nodes)
- Simple structure, complexity of some operations (n nodes, m edges):
  - add_edge(v) O(1)
  - remove_edge(v) O(m)
  - remove_node(v) O(m)
  - adjacent(u,v) O(m)
  - neighbors(v) O(m)

Adjacency list

- We keep simple lists of nodes and their neighbors
- Complexity of some operations (assuming an array-based implemementatoin):
  - add_node(v) O(1)
  - remove_node(v) O(m)
  - adjacent(u,v) O(n + min(deg(u), deg(v)))
  - neighbors(v) O(n + deg(v))

Adjacency matrix

	A	B	C	D
A		e	k	g
B			f	h
C				
D				

- We keep a $n \times n$ matrix
- Complexity of some operations:
  - add_node(v) O(n)
  - remove_node(v) O(n)
  - adjacent(u,v) O(1)
  - neighbors(v) O(n)

Interesting problems on graphs

- Is there a (directed) path between two nodes?
- What is the shortest path between two nodes?
- Is there a cycle in the graph?
- Is there a cycle that uses each edge exactly once? (Eulerian path)
- Is there a cycle that uses each node exactly once? (Hamiltonian path)
- Are all nodes of the graph connected?
- Is there a node that breaks the connectivity if removed?
- Is the graph planar: can it be drawn without crossing edges?
- Are two graphs isomorphic (have the same structure)?
- What is the importance of a web page, based on the links pointing to it?

Summary

- Graphs are data structures with many applications
- Reading on graphs: Goodrich, Tamassia, and Goldwasser (2013), chapter 14,

Next:
- Graph traversals
- Reading: Goodrich, Tamassia, and Goldwasser (2013), chapter 14)

Acknowledgments, credits, references

- The map on slide 2 is from OpenStreetMap, The other images are from Wikipedia, except the infectious disease graph which comes from Thurner et al. (2020).

Goodrich, Michael T., Roberto Tamassia, and Michael H. Goldwasser (2013). *Data Structures and Algorithms in Python.* John Wiley & Sons, Incorporated. isbn: 9781118476734.